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Ideal class semigroups of overrings

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Abstract

Let R be an integral domain an let T be an overring of R. There is a canonical semigroup homomorphism between the ideal class semigroup of R and the ideal class semigroup of T. We investigate conditions under which this semigroup homomorphism is surjective and we apply the results we obtain to the study of overrings of Clifford regular domains. We recover some known results of Bazzoni and we prove in certain more general situations that the Clifford regular property is inherited by an overring. In particular, we prove that if R is a Clifford regular domain such that the integral closure of R is a fractional overring, then every overring of R is Clifford regular. We also characterize among Clifford regular domains the ones that are stable.

Keywords: Ideal class semigroup; Flat extension; Overring; Clifford regular domain; Stable domain

1. Introduction

All rings we consider here are commutative and have an identity element, denoted by 1. A ring R is called *local* if R has an unique maximal ideal and *semilocal* if R has finitely many maximal ideals.

Let R be an integral domain with fraction field $Q(R)$. Recall that R is said to have *finite character* if every nonzero element of R is contained in at most finitely many maximal ideals of R. Equivalently, R has finite character if and only if every nonzero ideal of R is contained in at most finitely many maximal ideals of R.

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For R-submodules A and B of $Q(R)$, $A:B$ is defined as follows:

$$A:B=\{q\in Q(R)\mid qB\subseteq A\}.$$

Recall that a *fractional ideal* I of R is an R-submodule of $Q(R)$ such that $dI\subseteq R$ for some nonzero element $d\in R$. Equivalently, an R-submodule I of $Q(R)$ is a fractional ideal of R if and only if $R:I\neq 0$. Denote by $\mathfrak{F}(R)$ the semigroup of nonzero fractional ideals of R with the usual multiplication.

For $I\in\mathfrak{F}(R)$, let $[I]$ be the isomorphism class of I and recall that if $I,J\in\mathfrak{F}(R)$, then $I\simeq J\Leftrightarrow I=qJ$ for some $0\neq q\in Q(R)$.

Definition 1.1. The *ideal class semigroup* $S(R)$ of R consists of all isomorphism classes of nonzero fractional ideals of R, under the composition rule $[I]\cdot[J]=[IJ]$.

Note that $S(R)$ is a commutative semigroup with identity $[R]$.

While ideal class groups of integral domains have received a lot of attention, the investigation of ideal class semigroups has only recently begun. The natural points of departure for studying ideal class semigroups are the valuation domains and their global versions, the Prüfer domains. Bazzoni and Salce in [1] study ideal class semigroups of valuation domains and later Bazzoni in [2] studies the structure of ideal class semigroups of Prüfer domains. Ideal class semigroups of orders in number fields are investigated by Zanardo and Zannier in [21].

An element $[L]\in S(R)$ is called *idempotent* if $[L]^2=[L]$. Thus $[L]$ is idempotent if and only if $L=qL^2$ for some nonzero element $q\in Q(R)$. Note that if $[L]\in S(R)$ is idempotent, then there exists an idempotent fractional ideal $K\in\mathfrak{F}(R)$ such that $[K]=[L]$. Indeed, if $L=qL^2$, let $K=qL$ and observe that K and L are isomorphic fractional ideals and $K^2=q^2L^2=q(qL^2)=qL=K$. Thus, if $[L]\in S(R)$ is idempotent, we may assume that L itself is idempotent.

For an idempotent element $[L]\in S(R)$, we define

$$G_{[L]}=\{[IL]\mid [IJL]=[L]\text{ for some }[J]\in S(R)\}.$$

It is easy to see that $G_{[L]}$ is an abelian group with identity $[L]$. $G_{[L]}$ is called the *constituent group* of $S(R)$ associated to the idempotent $[L]$. Note that $G_{[L]}$ is the maximal subgroup of $S(R)$ having $[L]$ as identity element. Moreover, the constituent groups associated to distinct idempotents are disjoint.

An element $[I]\in S(R)$ is called *regular* if $[I]=[I]^2[X]$ for some $[X]\in S(R)$. Equivalently, $[I]$ is regular if and only if there exists a nonzero fractional ideal J of R such that $I=I^2J$.

By Lemma 1.1 of [2], $[I]$ is regular if and only if $I^2(I:I^2)=I$.

The regular elements of $S(R)$ form a subsemigroup of $S(R)$ which will be denoted by $\operatorname{Reg} S(R)$. Observe that

$$\operatorname{Reg} S(R)=\bigcup_{[L]}G_{[L]}$$

where $[L]$ runs through the set of all idempotents of $S(R)$.

Definition 1.2. The ideal class semigroup of R is said to be a *Clifford semigroup* if every element of $S(R)$ is regular.

Equivalently, $S(R)$ is a Clifford semigroup $\Leftrightarrow \operatorname{Reg} S(R) = S(R) \Leftrightarrow S(R)$ is a disjoint union of abelian groups.

Definition 1.3. An integral domain R is said to be *Clifford regular* if the ideal class semigroup of R is a Clifford semigroup.

Clifford regular domains are introduced by Zanardo and Zannier in [21] and Bazzoni and Salce in [1]. Bazzoni in [3] is the first to write down the definition of Clifford regular domains and study them in greater detail.

By Lemma 1.1 of [2], R is a Clifford regular domain $\Leftrightarrow I^2(I : I^2) = I$ for all $I \in \mathfrak{F}(R) \Leftrightarrow I^2(I : I^2) = I$ for every nonzero ideal I of R.

Examples of Clifford regular domains include valuation domains and Prüfer domains of finite character (see [1] and [2]). Bazzoni proves a number of properties of Clifford regular domains and completely characterizes the Noetherian Clifford regular domains and integrally closed Clifford regular domains. An integrally closed domain is Clifford regular if and only if it is a Prüfer domain of finite character [3, Theorem 4.5].

Definition 1.4. A subring T of $Q(R)$ such that $R \subseteq T$ is called an *overring* of R. If, in addition, T is a fractional ideal of R, then T is called a *fractional overring* of R.

For a nonzero fractional ideal I of R, it is well known that the endomorphism ring of I is canonically isomorphic to $I : I$. The fractional ideal $I : I$ is the largest overring of R in which I is an ideal. Moreover, an overring T of R is a fractional overring if and only if T is the endomorphism ring of a nonzero ideal of R.

Recall that $I \in \mathfrak{F}(R)$ is called *stable* if I is an invertible ideal of $I : I$. An integral domain R is called a *stable* domain if every nonzero fractional ideal of R is stable. Following Olberding's terminology of [14], an integral domain R is called *finitely stable* if every nonzero finitely generated fractional ideal of R is stable.

Bazzoni proves that the class of Clifford regular domains is properly intermediate between the class of stable domains and the class of finitely stable domains [3, Propositions 2.2 and 2.3] and concludes that a Noetherian domain is Clifford regular if and only if it is a stable domain. Therefore, the Noetherian case is well-understood. Characterizations of Noetherian stable domains can be found in [7,13,17,19,20]. We mention that the fact that Noetherian stable domains, or at least the Bass domains, are Clifford regular has been stated rather explicitly (without the terminology) in earlier work of L.S. Levy and R. Wiegand (page 3 of [11] and page 51 of [12]).

We describe in Proposition 2.2 the constituent group associated to an idempotent of $S(R)$. We then use this to give a converse to Proposition 2.2 of [3], thus characterizing the Clifford regular domains that are stable. In Theorem 2.6 we prove that an integral domain R is stable if and only if it is Clifford regular and every nonzero idempotent fractional ideal of R is a ring. As a corollary, we recover the structure of Noetherian Clifford regular domains proved by Bazzoni in [3, Theorem 3.1].

In Section 3 we consider for an overring T of R the canonical semigroup homomorphism $\phi_T^R : \mathfrak{F}(R) \to \mathfrak{F}(T)$ defined by $\phi_T^R(I) = IT$ for every $I \in \mathfrak{F}(R)$. We show that ϕ_T^R is surjective in either of the following cases:

(i) T is a fractional overring of R;
(ii) T is a flat overring of R;

(iii) T is well-centered on R;
(iv) T is a Noetherian overring of R.

We use Zariski's Main Theorem to deduce in Corollary 3.11 that if R is an integral domain such that the integral closure of R is a Prüfer domain and a fractional overring of R, then the canonical homomorphism ϕ_T^R is surjective for every overring T of R.

For a Noetherian domain R, we also conclude in Remark 3.12 that the map ϕ_T^R is surjective for every overring T of R if and only if $\dim R \leqslant 1$.

In Section 4 we study overrings of Clifford regular domains. By applying the results obtained in Section 3, we recover some known results of Bazzoni on fractional overrings and localizations of Clifford regular domains [3, Lemmas 2.4 and 2.5]. In addition, we prove in Corollary 4.4 that if T is a flat or well-centered overring of a Clifford regular domain R, then T is also Clifford regular.

We ask in Question 4.5 if every overring of a Clifford regular domain is again Clifford regular. By work of Bazzoni and Olberding, it follows that Question 4.5 has a positive answer if R is Noetherian or integrally closed. For a Clifford regular domain R, we are able to answer Question 4.5 in the affirmative if the integral closure of R is a fractional overring of R. The main result of Section 4 states that if R is a Clifford regular domain such that the integral closure $\overline{R}$ of R is a fractional overring of R, then every overring of R is Clifford regular. We prove that if R is a local finitely stable domain whose integral closure $\overline{R}$ has more than one maximal ideal, then $\overline{R}$ is a finitely generated R-module. We conclude that the Clifford regularity is inherited by every overring of a local Clifford regular domain whose integral closure is not a valuation domain.

2. Stable domains and Clifford regular domains

Let R be an integral domain and let $I \in \mathfrak{F}(R)$ such that $[I]$ is a regular element of $S(R)$. Let $J \in \mathfrak{F}(R)$ with $I = I^2J$ and note that IJ is idempotent and $[I] \in G_{[IJ]}$. Bazzoni gives in [3, Proposition 1.2] a description of the idempotent of $S(R)$ associated to $[I]$ which is independent of the choice of J:

Proposition 2.1. *With the above notations, let $E = I : I$ and $L = I(E : I)$. Then the following statements hold*:

(i) *$IJ = L$ and $[L]$ is the idempotent of $S(R)$ associated to $[I]$.*
(ii) *L is an idempotent ideal of E and $IL = I$.*
(iii) *$E = L : L = E : L$.*

We use Proposition 2.1 to give an explicit characterization of the group associated to an idempotent of $S(R)$.

Proposition 2.2. *Let L be an idempotent nonzero fractional ideal of R and set $D = L : L$. Then the subgroup $G_{[L]}$ of $S(R)$ associated to the idempotent $[L]$ of $S(R)$ is given by*

$$G_{[L]} = \{[K] \mid K \in \mathfrak{F}(R),\ KL = K \text{ and } K(D : K) = L\}.$$

Proof. Let $[K] \in G_{[L]}$, $K \in \mathfrak{F}(R)$. Then there are $I, J \in \mathfrak{F}(R)$ such that $[K] = [I][L]$ and $[I][J][L] = [L]$. By modifying I, J, if necessary, we may assume that $I, J \in \mathfrak{F}(R)$ satisfy $K = IL$ and $IJL = L$, i.e. $KJ = L$.

First note that $KL = K$. Indeed $KL = (IL)L = IL^2 = IL = K$. Now recall that $G_{[L]} \subseteq \operatorname{Reg} S(R)$, so $[K]$ is a regular element of $S(R)$ and we have $K^2(K : K^2) = K$.

As in the previous proposition, set $E = K : K$ and $T = K(E : K)$. Then $[T]$ is the idempotent of $S(R)$ associated to $[K]$. Hence $[L] = [T]$, so $T = qL$ for some nonzero $q \in Q(R)$. By Proposition 2.1 $K = KT = KqL$ and $K = KL$ imply $K = qK$. So $L = KJ = qKJ = qL = T$ and thus $D = E$. Hence $K(D : K) = L$ and the inclusion "$\subseteq$" is now proved.
Conversely, if K satisfies $KL = K$ and $K(D : K) = L$, then $[K] = [K][L]$ and $[K][D : K][L] = [L][L] = [L^2] = [L]$. So, by definition, $[K] \in G_{[L]}$. □

Remark 2.3. If $K \in \mathfrak{F}(R)$ satisfies $KL = K$ and $K(D : K) = L$, where L is idempotent and $D = L : L$, then $K : K = D$.

Proof. Let $x \in D = L : L$, so $xL \subseteq L$. Hence $xKL \subseteq KL$, so $xK \subseteq K$, i.e. $x \in K : K$. So $D \subseteq K : K$.

Now observe that L is an ideal of $K : K$, for if $x \in L$, $y \in K : K$, then $xy \in K(D : K)(K : K) = K(D : K) = L$. Since $D = L : L$ is the largest overring of R containing L as an ideal, we must have $K : K \subseteq D$. So $K : K = D$. □

We exhibit below two corollaries to Proposition 2.2.

Corollary 2.4. *Let L be a fractional overring of R. Then $[L]$ is an idempotent of $S(R)$ and the subgroup $G_{[L]}$ of $S(R)$ associated to $[L]$ coincides with the ideal class group $\mathcal{C}(L)$ of L. In particular, if L is semilocal, then $G_{[L]}$ is trivial.*

Proof. Since L is a fractional overring of R, clearly $[L]$ is an idempotent of $S(R)$ and we have $L : L = L$.

We show that $G_{[L]} = \mathcal{C}(L)$ by double inclusion, using Proposition 2.2.

"$\subseteq$" Let $[K] \in G_{[L]}$, so $K \in \mathfrak{F}(R)$, $KL = K$ and $K(L : K) = L$. Since $K \in \mathfrak{F}(R)$, we have that $KL \in \mathfrak{F}(L)$, so $[K] = [KL] \in \mathcal{C}(L)$. As $K(L : K) = L$, K is an invertible ideal of L, so $[K] \in \mathcal{C}(L)$.

"$\supseteq$" Let $[K] \in \mathcal{C}(L)$. So $K \in \mathfrak{F}(L)$ and $K(L : K) = L$. Since $K \in \mathfrak{F}(L)$, it follows that $KL = K$. Also, $K \in \mathfrak{F}(R)$, since K is a fractional ideal of L and L is a fractional ideal of R. Thus $[K] \in G_{[L]}$.

The last statement is obvious, since every invertible ideal in a semilocal ring is principal [5, Chapter I, Proposition 2.5]. □

A nonzero fractional ideal I of R is called *archimedean* if $I : I = R$. If R is a valuation domain with nonprincipal maximal ideal, then the isomorphism classes of nonprincipal fractional archimedean ideals of R form an abelian group under the usual multiplication of $S(R)$ [5, Chapter II, Theorem 4.10]. This group is denoted by $\operatorname{Arch}(R)$ and is called the archimedean group of R. For a description of $\operatorname{Arch}(R)$ in terms of the value group of R see [5, Chapter II, Proposition 4.12].

Bazzoni and Salce prove that if L is a nonzero idempotent prime ideal of a valuation domain R, then the group $G_{[L]}$ is precisely the archimedean group of the valuation domain R_L [1, Theorem 3]. We now use Proposition 2.2 to give a different proof for Bazzoni and Salce's result.

Corollary 2.5. *Let R be a valuation domain and let L be a nonzero idempotent prime ideal of R. Then $G_{[L]} = \mathrm{Arch}(R_L)$.*

Proof. "$\subseteq$" Let $[K] \in G_{[L]}$. Note that $L : L = R_L$, hence by Remark 2.3 $K : K = R_L$. So K is an archimedean fractional ideal of R_L. Clearly K is a nonprincipal fractional ideal of R_L, otherwise $KL = K$ implies $LR_L = R_L$. Thus, $[K] \in \mathrm{Arch}(R_L)$.

"$\supseteq$" Let $[K] \in \mathrm{Arch}(R_L)$. We may assume that K is a nonprincipal archimedean ideal of R_L. Since R is a valuation domain, R_L is a fractional ideal of R, hence $K \in \mathfrak{F}(R)$. Since $LR_L = L$ and L is idempotent, the maximal ideal L of the valuation domain R_L is nonprincipal, and hence $KL = K$. Since $[L]$ is the unit of $\mathrm{Arch}(R_L)$, $[K][R_L : K] = [L]$, so there is $\lambda \in R_L$ such that $K(R_L : K) = \lambda L$. But $K(R_L : K)$ is idempotent by Proposition 2.1, hence $K(R_L : K) = L$. Proposition 2.2 now shows that $[K] \in G_{[L]}$. $\square$

We now use Proposition 2.2 to give a new characterization of stable domains:

Theorem 2.6. *An integral domain R is a stable domain if and only if R is Clifford regular and every nonzero idempotent fractional ideal of R is a ring.*

Proof. "$\Rightarrow$" If R is stable, then R is Clifford regular [3, Proposition 2.2]. Now let I be a nonzero idempotent fractional ideal of R and let $E = I : I$. Since I is stable, we have $E = I(E : I) = I((I : I) : I) = I(I : I^2) = I(I : I) = I$. Thus $I = E$, so I is a ring.

"$\Leftarrow$" Let I be a nonzero ideal of R. We want to show that I is stable. Since $S(R)$ is a Clifford semigroup, $[I] \in G_{[L]}$ for some idempotent element $[L]$ of $S(R)$. Note that we may assume that L itself is idempotent, and hence, by hypothesis, L is a ring and so $D := L : L = L$. Since $[I] \in G_{[L]}$, we know that $I : I = L : L = D$ and $I(D : I) = L = D$. This shows that I is invertible as an ideal of $D = I : I$, so I is stable. Thus, R is a stable domain. $\square$

Note that if R is Noetherian and I is a nonzero idempotent fractional ideal of R, then I is a finitely generated idempotent ideal of $I : I$. Thus $I = I : I$, so I is a ring. Thus, we get the following corollary:

Corollary 2.7. *A Noetherian domain R is stable if and only if it is Clifford regular.*

3. Ideal class semigroups of overrings

Throughout this section, R will be an integral domain and T an overring of R. If I is a nonzero fractional ideal of R, then

$$0 \neq R : I \subseteq T : IT$$

thus IT is a nonzero fractional ideal of T and there is a canonical map $\phi_T^R : \mathfrak{F}(R) \to \mathfrak{F}(T)$ defined by $\phi_T^R(I) = IT$ for every $I \in \mathfrak{F}(R)$. It is easy to see that ϕ_T^R is actually a semigroup homomorphism.

Note that ϕ_T^R induces a map $\overline{\phi_T^R} : S(R) \to S(T)$ given by $\overline{\phi_T^R}([I]) = [IT]$ for every $I \in \mathfrak{F}(R)$. $\overline{\phi_T^R}$ is also a semigroup homomorphism. Note that $\overline{\phi_T^R}$ maps regular elements of $S(R)$ to regular elements of $S(T)$.

Proposition 3.1. *With the above notations, the following statements hold*:

(i) $\mathrm{Ker}(\overline{\phi_T^R}) = \{[I] \mid \exists 0 \neq \lambda \in Q(R)$ *such that* $\lambda I \in \mathrm{Ker}(\phi_T^R)\}$. *In particular, if* $I \in \mathrm{Ker}(\phi_T^R)$, *then* $[I] \in \mathrm{Ker}(\overline{\phi_T^R})$.
(ii) *If T is a proper overring of R, then ϕ_T^R has a nontrivial kernel.*

Proof. (i) If $[I] \in \mathrm{Ker}(\overline{\phi_T^R})$, then $[IT] = [T]$, so $\exists 0 \neq \lambda \in Q(R)$ such that $\lambda IT = T$. Hence $(\lambda I)T = T$, so $\lambda I \in \mathrm{Ker}(\phi_T^R)$. Conversely, if $\lambda I \in \mathrm{Ker}(\phi_T^R)$ for some $0 \neq \lambda \in Q(R)$, then $\overline{\phi_T^R}([I]) = [IT] = [\lambda IT] = [T]$, so $[I] \in \mathrm{Ker}(\overline{\phi_T^R})$.

(ii) Since T is a proper overring of R, there exists a nonzero element $\lambda \in T \setminus R$. Let $I = R + R\lambda$. Then I is a fractional ideal of R, $I \neq R$ and $IT = T + T\lambda = T$, so $I \in \mathrm{Ker}(\phi_T^R)$. Hence ϕ_T^R has a nontrivial kernel. □

Remark 3.2. For a proper overring T of R, it can happen that the canonical map $\overline{\phi_T^R}$ has trivial kernel. In fact, if R is a PID, then $\overline{\phi_T^R}$ is an isomorphism for every overring T of R. (Note that the ideal class semigroup of a PID is trivial and that every overring of a PID is again a PID.)

Remark 3.3.

(i) ϕ_T^R is surjective if and only if every fractional ideal of T is the extension of a fractional ideal of R.
(ii) $\overline{\phi_T^R}$ is surjective if and only if ϕ_T^R is surjective.

The next results give sufficient conditions for the surjectivity of ϕ_T^R:

Proposition 3.4. *If T is a fractional overring of R, then ϕ_T^R is surjective. In particular, if T is a finitely generated R-module, then ϕ_T^R is surjective.*

Proof. By hypothesis, there exists $0 \neq d \in R$ such that $dT \subseteq R$, so $T \subseteq \frac{1}{d}R$.

Let X be a nonzero fractional ideal of T. Then X is an R-submodule of $Q(R)$ and $0 \neq T : X \subseteq \frac{1}{d}R : X = \frac{1}{d}(R : X)$, so $R : X \neq 0$. Hence X is a fractional ideal of R. Since T contains 1, we have $X = XT$, showing that ϕ_T^R is surjective.

If T is a finitely generated R-module, then T is a fractional overring of R and, by above, ϕ_T^R is surjective. □

The proof of the following proposition is immediate.

Proposition 3.5. *If every ideal of T is the extension of an ideal of R, then ϕ_T^R is surjective.*

Proof. Let X be a nonzero fractional ideal of T. Then $\exists 0 \neq d \in T$ such that $dX = J$ is a nonzero ideal of T. By hypothesis, $J = IT$ for some nonzero ideal I of R. Hence $X = (1/d)J =$

$(1/d)IT$. Thus, X is the extension to T of the nonzero fractional ideal $(1/d)I$ of R. Hence ϕ_T^R is surjective. □

If R is a Prüfer domain and T is an overring of R, then every ideal of T is extended from R [5, p. 95]. Hence, if R is a Prüfer domain, then the canonical map ϕ_T^R is surjective for every overring T of R.

Remark 3.6. Even if ϕ_T^R is surjective it does not follow, in general, that every ideal of T is the extension of an ideal of R. For example, if R is a local domain with maximal ideal m and T is a fractional integral overring of R with at least two maximal ideals, then ϕ_T^R is surjective (by Proposition 3.4), but no maximal ideal of T is extended from R. Indeed, assume that M is a maximal ideal of T which is extended from R, say $M = IT$ for some nonzero ideal I of R. Clearly I is a proper ideal of R, so $I \subseteq m$. Since $M \cap R = m$, we have $M = IT \subseteq mT \subseteq M$, hence $mT = M$. Now let N be a maximal ideal of T, distinct from M. Since $N \cap R = m$, we have $M = mT \subseteq N$, hence $M = N$, contradiction.

Definition 3.7. Let R be an integral domain and let T be an overring of R.

(1) T is called a *localization* of R if $T = S^{-1}R = R_S$, where S is a multiplicatively closed subset of nonzero elements of R.
(2) T is called a *flat overring* of R if T is a flat R-module.
(3) T is said to be *well-centered on* R if for each $t \in T$ there exists a unit $u \in T$ such that $ut = r \in R$. Thus, T is well-centered on R if and only if each principal ideal of T is generated by an element of R.

Flat overrings are considered in [16] and [10, Chapter IV]. Well-centered overrings of an integral domain are introduced and studied by Heinzer and Roitman in [8]. By Proposition 4.14 of [10], an overring T of an integral domain R is a flat overring if and only if $T_M = R_{M\cap R}$ for all maximal ideals M of T.

Note that a localization of R is both flat over R and well-centered on R.

Proposition 3.8. *If T is flat over R or well-centered on R, then ϕ_T^R is surjective. In particular, if T is a localization of R, then ϕ_T^R is surjective.*

Proof. We show that in either case, every ideal of T is extended from R. The conclusion then follows from Proposition 3.5.

Assume that T is a flat overring of R. Let J be a nonzero ideal of T. Note first that $J \cap R \neq 0$. For each maximal ideal M of T we have $JT_M = JR_{M\cap R} = (J \cap R)R_{M\cap R} = ((J \cap R)T)T_M$. Thus, $J = (J \cap R)T$, so J is the extension of a nonzero ideal of R.

If T is well-centered on R, then every principal ideal of T is the extension of a principal ideal of R. Hence every nonzero ideal of T is the extension of a nonzero ideal of R. □

Proposition 3.9. *If T is a Noetherian overring of R, then ϕ_T^R is surjective.*

Proof. Let J be a nonzero fractional ideal of T. Since T is Noetherian, J is a finitely generated T-module. So there exist nonzero elements $d, a_1, a_2, \ldots, a_n$ of R such that $J = T\frac{a_1}{d} + T\frac{a_2}{d} +$

$\cdots + T\frac{a_n}{d}$. If $I = \frac{1}{d}(Ra_1 + Ra_2 + \cdots + Ra_n)$, then I is a nonzero fractional ideal of R and $J = IT$. Thus, ϕ_T^R is surjective. $\square$

If R is a subring of a ring T and P is a prime ideal of T, then P is said to be *isolated* over $R \cap P$ if P is maximal and minimal with respect to the primes of T whose intersection with R is $R \cap P$. We recall now the following variant of Zariski's Main Theorem, due to Peskine and Evans (see [4] and [15]):

Zariski's Main Theorem. *Let R be a subring of T such that R is integrally closed in T and there exist $t_1, t_2, \dots, t_n \in T$ with T integral over $R[t_1, t_2, \dots, t_n]$. If a prime ideal P of T is isolated over $P \cap R$, then there exists an $s \in R \setminus P \cap R$ such that $T_s = R_s$.*

Zariski's Main Theorem is the main ingredient in proving the following result:

Proposition 3.10. *Let R be an integral domain such that $\overline{R}$ is a Prüfer domain. Let T be an overring of R. Then T is a flat extension of $\overline{R} \cap T$.*

Proof. Let $A = \overline{R} \cap T$. Let B be a finitely generated A-subalgebra of T and note that A is integrally closed in B. Let $p \subsetneq q$ be two prime ideals of B such that $p \cap A = q \cap A = m$. By the Going-Up Theorem, there exist prime ideals $P \subsetneq Q$ of $\overline{B}$ lying over p and q, respectively. Note that $\overline{A} = \overline{R}$ is a Prüfer domain, so $P = (P \cap \overline{A})\overline{B}$ and $Q = (Q \cap \overline{A})\overline{B}$. Then $P \cap \overline{A} \subsetneq Q \cap \overline{A}$ are two distinct comparable prime ideals of $\overline{A}$ lying over the prime ideal m of A, contradiction. Thus, there do not exist two prime ideals of B with one properly contained in the other that have the same contraction to A. Hence, every prime ideal of B is isolated over $P \cap A$. By Zariski's Main Theorem, it follows that for every prime ideal P of B, there exists $s \in A \setminus P \cap A$ such that $B_s = A_s$. Hence $B_P = A_{P \cap A}$, and by Proposition 4.14 of [10], it follows that B is flat over A. So every finitely generated A-subalgebra of T is flat over A. Since T is the direct limit of its finitely generated A-subalgebras, it follows that T is flat over A. $\square$

Corollary 3.11. *Let R be an integral domain such that $\overline{R}$ is a Prüfer domain and a fractional overring of R. Then the canonical map $\phi_T^R : \mathfrak{F}(R) \to \mathfrak{F}(T)$ is surjective for every overring T of R.*

Proof. Let T be an overring of R and let $A = \overline{R} \cap T$. Since $\overline{R}$ is a fractional overring of R and $A \subseteq \overline{R}$, A is a fractional overring of R, so the canonical map $\phi_A^R : \mathfrak{F}(R) \to \mathfrak{F}(A)$ is surjective, by Proposition 3.4. By Proposition 3.10, T is a flat overring of A. Hence, by Proposition 3.8, the canonical map $\phi_T^A : \mathfrak{F}(A) \to \mathfrak{F}(T)$ is surjective. We conclude that $\phi_T^R = \phi_T^A \circ \phi_A^R$ is surjective, as a composition of two surjective maps. $\square$

We end this section observing that the canonical map $\phi_T^R : \mathfrak{F}(R) \to \mathfrak{F}(T)$ needs not be surjective, in general:

Remark 3.12. The image under ϕ_T^R of a finitely generated fractional ideal of R is a finitely generated fractional ideal of T. If R is a Noetherian domain of dim > 1, then R has a non-Noetherian valuation overring T [22, Chapter VI, §4, Theorem 5] and hence the map ϕ_T^R is not surjective. Moreover, by Krull–Akizuki's Theorem [9, Theorem 93], every overring of a 1-dimensional Noetherian domain is again Noetherian and of dimension at most 1. In view of the above and Proposition 3.9, we conclude that for a Noetherian domain R, the map ϕ_T^R is surjective for every overring T of R if and only if $\dim R \leqslant 1$.

4. Overrings of Clifford regular domains

In this section we apply the results of the previous section to the study of overrings of Clifford regular domains.

Bazzoni in [3] proved that fractional overrings and localizations of Clifford regular domains are Clifford regular. We recover here these results and we also prove in certain more general situations that the Clifford regular property is inherited by an overring.

Let R be an integral domain and let T be an overring of R. Recall that there is a semigroup homomorphism $\phi_T^R : \mathfrak{F}(R) \to \mathfrak{F}(T)$ defined by $\phi_T^R(I) = IT$ for every $I \in \mathfrak{F}(R)$.

Proposition 4.1. *If ϕ_T^R is surjective and R is Clifford regular, then T is Clifford regular.*

Proof. Since the induced map $\overline{\phi_T^R} : S(R) \to S(T)$ is also a surjective semigroup homomorphism and the homomorphic image of a Clifford semigroup is a Clifford semigroup, it follows that T is a Clifford regular domain. □

Corollary 4.2. *Let R be a Clifford regular domain. If T is a fractional overring of R, then T is Clifford regular and ϕ_T^R is surjective.*

Proof. Follows at once from Proposition 3.4 and the previous result. □

Corollary 4.3. *Let R be a Clifford regular domain and let T be an overring of R such that every ideal of T is the extension of an ideal of R. Then T is Clifford regular.*

Proof. Follows at once from Propositions 3.5 and 4.1. □

Corollary 4.4. *Let R be a Clifford regular domain and let T be an overring of R.*

(i) *If T is a flat overring of R, then T is Clifford regular and ϕ_T^R is surjective.*
(ii) *If T is well-centered on R, then T is Clifford regular and ϕ_T^R is surjective.*

In particular, if T is a localization of R, then T is Clifford regular.

Proof. Follows immediately from Propositions 3.8 and 4.1. □

We do not know if the Clifford regularity is inherited by every overring of a Clifford regular domain. We ask the following question:

Question 4.5. If R is a Clifford regular domain, is every overring of R again Clifford regular?

We observe that Question 4.5 has a positive answer if R is Noetherian or integrally closed. In the Noetherian case, this follows from Corollary 2.7 and the fact that every overring of a stable domain is again stable [14, Theorem 5.1]. If R is integrally closed, then R is a Prüfer domain of finite character and thus the same holds for every overring of R. Since Prüfer domains of finite character are Clifford regular, it follows that every overring of R is Clifford regular.

We now give a positive answer to Question 4.5 in the case the integral closure of R is a fractional overring of R.

Theorem 4.6. *Let R be a Clifford regular domain such that the integral closure $\overline{R}$ of R is a fractional overring of R. Then every overring of R is Clifford regular.*

Proof. By Corollary 4.2, a fractional overring of a Clifford regular domain is again Clifford regular, hence $\overline{R}$ is Clifford regular. But $\overline{R}$ is integrally closed, so $\overline{R}$ is a Prüfer domain.

Let T be an overring of R. By Corollary 3.11, the canonical map $\phi_T^R : \mathfrak{F}(R) \to \mathfrak{F}(T)$ is surjective. Proposition 4.1 now shows that T is Clifford regular. $\square$

We now show that a local Clifford regular domain whose integral closure is not a valuation domain satisfies the hypotheses of Theorem 4.6.

Recall that if R is a finitely stable domain, the integral closure $\overline{R}$ of R is a Prüfer domain and every R-submodule of $\overline{R}$ containing R is a ring [18, Proposition 2.1]. Moreover, if R is local, then $\overline{R}$ has at most three maximal ideals [14, Proposition 2.3]. The author wishes to thank B. Olberding for his help in proving the following result:

Proposition 4.7. *Let R be a local finitely stable domain such that the integral closure $\overline{R}$ of R is not a valuation domain. Then $\overline{R}$ is a finitely generated R-module.*

Proof. We first prove the existence of a module-finite overring T of R such that T is contained in $\overline{R}$ and T and $\overline{R}$ have the same number of maximal ideals.

Since $\overline{R}$ is not a valuation domain, $\overline{R}$ has either two or three maximal ideals. If $\overline{R}$ has two maximal ideals m_1 and m_2, let $x \in m_1 \setminus m_2$ and set $T = R + Rx$. If $\overline{R}$ has three maximal ideals m_1, m_2, and m_3, then choose $x \in m_1 \setminus (m_2 \cup m_3)$ and $y \in m_2 \setminus (m_1 \cup m_3)$ and set $T = R + Rx + Ry$. In both cases, $T \subseteq \overline{R}$ is a module-finite overring of R and the maximal ideals of T are precisely the restrictions to T of the maximal ideals of $\overline{R}$. So T and $\overline{R}$ have the same number of maximal ideals.

Now let m be the maximal ideal of R and choose T as above. Note that every overring S of R such that $T \subseteq S \subseteq \overline{R}$ also has the same number of maximal ideals as T and $\overline{R}$. Since R is finitely stable, it follows that T/mT is a finite-dimensional R/m-algebra with the property that every R/m-subalgebra containing the identity is a ring. Hence Handelman's Lemma [6, Lemma 5] applies and since T/mT has more than one maximal ideal, it follows that $T/mT = R/m \times R/m$ or $T/mT = R/m \times R/m \times R/m$. Now if T is not equal to $\overline{R}$, we can choose a pair of distinct module-finite overrings T_1 and T_2 of R, such that T_1 and T_2 are contained in $\overline{R}$ and have the same number of maximal ideals as $\overline{R}$. If $T_1 \subset T_2$, since T_1/mT_1 and T_2/mT_2 are necessarily isomorphic by Handelman's Lemma, it follows that $T_1 = T_2$, contradiction. If $T_1 \nsubseteq T_2$, then $T_2 \subset T_1T_2$, and the above argument applied to T_2 and T_1T_2 shows that $T_2 = T_1T_2$, and hence $T_1 \subseteq T_2$, contradiction. Thus $T = \overline{R}$, so $\overline{R}$ is a finitely generated R-module. $\square$

Since Clifford regular domains are finitely stable, Theorem 4.6 and Proposition 4.7 yield the following corollary:

Corollary 4.8. *Let R be a local Clifford regular domain such that the integral closure of R is not a valuation domain. Then every overring of R is Clifford regular.*

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References

[1] S. Bazzoni, L. Salce, Groups in the class semigroups of valuation domains, Israel J. Math. 95 (1996) 135–155.
[2] S. Bazzoni, Class semigroups of Prüfer domains, J. Algebra 184 (1996) 613–631.
[3] S. Bazzoni, Clifford regular domains, J. Algebra 238 (2001) 703–722.
[4] E.G. Evans Jr., A generalization of Zariski's main theorem, Proc. Amer. Math. Soc. 26 (1970) 45–48.
[5] L. Fuchs, L. Salce, Modules over Non-Noetherian Domains, Math. Surveys Monogr., vol. 84, 2001.
[6] D. Handelman, Propinquity of one-dimensional Gorenstein rings, J. Pure Appl. Algebra 26 (1982) 145–150.
[7] W.J. Heinzer, D. Lantz, K. Shah, The Ratliff–Rush ideals in a Noetherian ring, Comm. Algebra 20 (2) (1992) 591–622.
[8] W.J. Heinzer, M. Roitman, Well-centered overrings of an integral domain, J. Algebra 272 (2004) 435–455.
[9] I. Kaplansky, Commutative Rings, Allyn and Bacon Inc., Boston, MA, 1970.
[10] M.D. Larsen, P.J. McCarthy, Multiplicative Theory of Ideals, Pure Appl. Math., vol. 43, Academic Press, New York, 1971.
[11] L.S. Levy, Modules over Dedekind-like rings, J. Algebra 93 (1985) 1–116.
[12] L.S. Levy, R. Wiegand, Dedekind-like behavior of rings with 2-generated ideals, J. Pure Appl. Algebra 37 (1985) 41–58.
[13] J. Lipman, Stable ideals and Arf rings, Amer. J. Math. 93 (1971) 649–685.
[14] B. Olberding, On the structure of stable domains, Comm. Algebra 30 (2) (2002) 877–895.
[15] C. Peskine, Une généralisation du "main theorem" de Zariski, Bull. Sci. Math. (2) 90 (1966) 119–127.
[16] F. Richman, Generalized quotient rings, Proc. Amer. Math. Soc. 16 (1965) 794–799.
[17] D.E. Rush, Rings with two-generated ideals, J. Pure Appl. Algebra 73 (1991) 257–275.
[18] D.E. Rush, Two-generated ideals and representations of abelian groups over valuation rings, J. Algebra 177 (1995) 77–101.
[19] J. Sally, W. Vasconcelos, Stable rings and a problem of Bass, Bull. Amer. Math. Soc. 79 (1973) 574–576.
[20] J. Sally, W. Vasconcelos, Stable rings, J. Pure Appl. Algebra 4 (1974) 319–336.
[21] P. Zanardo, U. Zannier, The class semigroup of orders in number fields, Math. Proc. Cambridge Philos. Soc. 115 (1994) 379–391.
[22] O. Zariski, P. Samuel, Commutative Algebra, vol. II, van Nostrand, 1960.